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An Inversion Formula for the Distributional Generalised Laplace Transformation

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A unification and generalisation of several extensions of the classical Laplace transform was given by Professor H. M. Srivastava in 1968. S. K. Sinha gave complex inversion and Tauberian theorems for this general integral transformation. A. K. Tiwari and A. Ko gave several further properties in the distributional sense. S. K. Akhaury extended the generalisation to distributions and derived the Abelian theorems of the initial and final value types. In the present paper we give the inversion formula for the distributional generalised Laplace transformation.

1. INTRODUCTION AND DEFINITIONS

A unification and generalisation of several interesting extensions of the classical Laplace transform:

$$L[f(t), s] = \int_0^\infty e^{-st} f(t)\, dt, \qquad \mathrm{Re}(s) > 0 \tag{1.1}$$

was given by Srivastava [6] in the form (see also Srivastava *et al.* [7]):

$$S_{q,k,m}^{(\rho,\sigma)}[f(t), s] = \int_0^\infty (st)^{\sigma-1/2} e^{-qst/2} W_{k,m}(\rho st) f(t)\, dt, \tag{1.2}$$

where $W_{k,m}(z)$ denotes the Whittaker function of the second kind (Whittaker and Watson [9, p. 339]). In the present paper we extend to Schwartz's distribution an inversion formula for the generalised Laplace transform (1.2).

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We consider the inversion formula for the generalisation [cf. Eq. (1.2)]

$$F(s) = \int_0^\infty e^{-qst/2}(st)^{\sigma-1/2} W_{k,m}(\rho st) f(t)\, dt, \tag{1.3}$$

where $f(t)$ is locally integrable and $f(t) = 0$ for $-\infty < t < 0$.

Differential operators U_n, V_n for the generalised Laplace transform (1.3) are defined as

$$\begin{aligned}
&U_0[F(x)] = xF(x)\\
&U_1[F(x)] = (-1)x^{m-k-5/2}Dx^{k-m+1/2}F(x) \ldots\\
&U_n[F(x)] = (-1)^n x^{m-k-n-3/2}D^n x^{k-m+1/2}F(x)\ (n = 2, 3, 4, \ldots),
\end{aligned}$$

where $D = x^2(d/dx)$ and

$$V_n[F(u)] = \frac{u^{-2}(n+k+m-1/2)^{3/2}}{\Gamma(n+k+m-1/2)}\left\{U_n\left[F\left(\frac{n+k+m-1/2}{u}\right)\right]\right\}$$

2. Notations

Suppose that I denotes an open interval on the real line. Let $\mathfrak{D}_I$ to be the space of smooth functions on I having compact support with respect to I. $\mathfrak{D}'_I$ is Schwartz's dual space of distributions on I. These spaces are simply denoted by $\mathfrak{D}$ and $\mathfrak{D}'$, if I is the entire real line. S_t and S'_t stand as the space of smooth functions of rapid descent and the space of tempered distributions on $-\infty < t < \infty$. $S_{u,t}$ denotes the space of rapidly descending smooth functions on the (u, t)-plane.

Let a and b $(a < b)$ be real numbers and let $K_{a,b}(t)$ be a positive smooth function on $-\infty < t < \infty$ such that

$$\begin{aligned}
K_{a,b}(t) &= e^{at}, \qquad 0 \le t < \infty,\\
&= e^{bt}, \qquad -\infty < t < 0.
\end{aligned}$$

The space $L_{a,b}$ denotes the space of all smooth functions $\phi(t)$ from R^1 into C^1 such that as $|t| \to \infty$, $|K_{a,b}(t) D_t^n \phi(t)| \to 0$. Now, we want to prove that $e^{-qst/2} \in L_{a,b}$ for $a \le \mathrm{Re}(s) \le b$. For this it is sufficient to show that

$$|K_{a,b}(t) D_t^n e^{-qst/2}| \to 0 \qquad \text{as } |t| \to \infty.$$

We have

$$|K_{a,b}(t)D_t^n e^{-qst/2}| = |K_{a,b}(t)(-qs/2)^n e^{-qst/2}|. \tag{2.1}$$

The right-hand side of (2.1) tends to zero as $|t| \to \infty$ for $a \leq \mathrm{Re}(s) \leq b$. This implies that

$$e^{-qst/2} \in L_{a,b} \qquad \text{for } a \leq \mathrm{Re}(s) \leq b.$$

LEMMA 2.1. *Let a, b, $\sigma \in R^n$ with $a \leq \sigma \leq b$. If $\psi \in s$, then $\{e^{-q\sigma t/2}\}\ \psi \in L_{a,b}$. If $\{\psi_\mu\}_{\mu=1}^{\infty}$ converges in S to zero, then $[\{e^{-q\sigma t/2}\}\psi_\mu]_{\mu=1}^{\infty}$ also converges in $L_{a,b}$ to zero.*

Proof. Proceeding as in Zemannian [10] it can be proved easily.

THEOREM 2.1. *If $f \in L'_{a,b}$, then $e^{-q\sigma t/2} f \in S$ for $a \leq \sigma \leq b$.*

Proof. It follows directly with the help of above lemma.

3. THE MAIN INVERSION FORMULA

THEOREM 3.1. *Let $f(u)$ be locally integrable and also let $f(u) = 0$ for $-\infty < u < 0$. Let I denote the interval $0 \leq u < \infty$. If $f(u)$ is a generalised Laplace transformable distribution with its support contained in I, then, in the sense of convergence in $\mathcal{D}'_I$,*

$$f(u) = \lim_{n\to\infty} \frac{u^{-2}(n+k+\sigma-1/2)^{3/2}}{\Gamma(n+k+\sigma-1/2)} U_n\left[F\left(\frac{n+k+\sigma-1/2}{u}\right)\right].$$

Proof. Let $\phi(u)$ be in $\mathcal{D}_I$ whose support is contained in $A \leq u \leq B$, where $0 < A < B < \infty$. Then we have

$$\langle V_n[F(u)], \phi(u)\rangle$$

$$= \left\langle \frac{u^{-2}(n+k+\sigma-1/2)^{3/2}}{\Gamma(n+k+\sigma-1/2)} U_n\left[F\left(\frac{n+k+\sigma-1/2}{u}\right)\right], \phi(u)\right\rangle$$

$$= \left\langle \frac{u^{-2}\gamma^{3/2}}{\Gamma(\gamma)} \cdot (-1)^n (\gamma/u)^{\sigma-k-n-3/2}\left[(\gamma/u)^2 \frac{d}{d(\gamma/u)}\right]^n (\gamma/u)^{k-\sigma+1/2} F(\gamma/u), \phi(u)\right\rangle,$$

where $n+k+\sigma-1/2=\gamma$,

$$= \left\langle F(\gamma/u), (-1)^n(\gamma/u)^{k-\sigma+1/2}\frac{d^n}{du^n}\left[\frac{\gamma^{n+3/2}u^{-2}}{\Gamma(\gamma)}(\gamma/u)^{\sigma-k-n-3/2}\phi(u)\right]\right\rangle$$

$$= \langle F(\gamma/u), \phi_n(u)\rangle, \tag{3.1}$$

where $\phi_n(u) = (-1)^n \gamma^{n+3/2}/\Gamma(\gamma))\ (\gamma/u)^{k-\sigma+1/2}\ (d^n/du^n)\ [u^{-2}(\gamma/u)^{\sigma-k-n-3/2}\phi(u)]$. $\phi_n(u)$ is in $\mathfrak{D}_I$ and its support is contained in $A \le u \le B$. Moreover, $F(\gamma/u)$ is a smooth function on $\sigma_1 < \gamma/u < \infty$. Thus, for all sufficiently large n, (3.1) is equal to

$$\langle \phi_n(u), \langle f(t), e^{-q(\gamma/u)t/2}((\gamma/u)t)^{\sigma-1/2} W_{k,m}(\rho(\gamma/u)t)\rangle\rangle \tag{3.2}$$

Let $G_f > 0$ be the lower bound on the support of f. Now we take two smooth functions $\zeta(t)$ and $\xi_n(\gamma/u)$ such that $\zeta(t)$ is equal to 1 on a neighbourhood of $G_f \le t < \infty$ and is identically zero for $-\infty < t < T$, where $0 < T < G_f$, and the smooth function $\xi_n(\gamma/u) \in \mathfrak{D}_I$ is equal to 1 on a neighbourhood of the support of $\phi_n(u)$ and is identically zero outside $A - \varepsilon' \le u \le B + \varepsilon'$ for some $\varepsilon' > 0$. Using these assumptions, (3.2) can be written in the form

$$\left\langle \phi_n(u), \left\langle e^{-q\lambda t/2} f(t), \zeta(t)\xi_n(\gamma/u) e^{-q(\gamma/u-2\lambda)t}\left(\frac{\gamma}{u}t\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\right\rangle\right\rangle, \tag{3.3}$$

$(\gamma/u - 2\lambda) > 0$ for fixed λ and all sufficiently large n on the support of $\xi_n(\gamma/u)$. Clearly, the testing function in (3.3) is in $S_{u,t}$. Moreover, $\phi_n(u) \in S_t'$ and, also, $e^{-q\lambda t/2}f(t) \in S_t'$ for sufficiently large λ (Zemannian [10]). Under these facts we can switch the inner product in (3.3) into

$$\left\langle \phi_n(u), \left\langle f(t), e^{-q(\gamma/u)t/2}\left(\frac{\gamma t}{u}\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\zeta(t)\right\rangle\right\rangle = \langle f(t), \zeta(t)\,\beta(t,n)\rangle \tag{3.4}$$

where $\beta(t, n) = \langle \phi_n(u), e^{-q(\gamma/u)t/2}\ ((\gamma/u)t)^{\sigma-1/2} W_{k,m}(\rho(\gamma/u)t)\rangle$.

The expression (3.4) has a sense, in fact, $f(t) \in L'_{a,b}$ for sufficiently large a and b and $\zeta(t)\beta(t, n) \in L'_{a,b}$ because

$$\begin{aligned} |D_t^r \beta(t,n)| &= \left| D_t^r \left\langle \phi_n(u), e^{-q(\gamma/u)t/2}\left(\frac{\gamma}{u}t\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\right\rangle\right| \\ &= \left|\left\langle \phi_n(u), D_t^r\left\{e^{-q(\gamma/u)t/2}\left(\frac{\gamma}{u}t\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\right\}\right\rangle\right|, \end{aligned}$$

where $D_t^r = (\partial^r/\partial t^r)\ (t > 0)$.

By using the results of Slater [5, p.25], we get

R.H.S.

$$= \left| \left\langle \phi_n(u), \left\{ \sum_{r=0}^{n} \sum_{\nu=0}^{r} \binom{n}{r} (-1)^{n-r+\nu} \left(\frac{1}{2} \overline{\frac{q}{\rho} - 1} \right)^{n-r} \rho^{n-r+\nu/2} \frac{(\sigma-m)!}{(\sigma-m-r+\nu)!} \right. \right. \right.$$

$$\left. \left. \left. \times \binom{r}{\nu} \left\{ e^{-q(\gamma/u)t/2} \left(\frac{\gamma}{u} \right)^n \left(\frac{\gamma}{u} t \right)^{\sigma-r+\nu/2-1/2} W_{k+\nu/2,m-\nu/2} \left(\rho \frac{\gamma}{u} t \right) \right\} \right\rangle \right|$$

$$= \left| \sum_{r=0}^{n} \binom{n}{r} \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{n-r+\nu} \left(\frac{1}{2} \overline{\frac{q}{\rho} - 1} \right)^{n-r} \rho^{n-r+\nu/2} \frac{(\sigma-m)!}{(\sigma-m-r+\nu)!} \right.$$

$$\left. \times \int_A^\beta \left(\frac{\gamma}{u} \right)^n \left\{ e^{-q(\gamma/u)t/2} \left(\frac{\gamma}{u} t \right)^{\sigma-r+\nu/2-1/2} W_{k+\nu/2,m-\nu/2} \left(\rho \frac{\gamma}{u} t \right) \right\} \phi_n(u)\, du \right|.$$

as the support of $\phi_n(u)$ is contained in $A \leq u \leq B$.

Now, since $e^{-q(\gamma/u)t/2}\,((\gamma/u)t)^{\sigma-r+\nu/2-1/2} W_{k+\nu/2,m-\nu/2}(\rho(\gamma/u)t)$ is finite at A and also at B and in between this interval the function is continuous, therefore it must be bounded within this interval. Let us consider that the upper bound of this function be attained at $u = \eta$, then

$$\left| D_t^r \beta(t,n) \right| \leq \left| \sum_{r=0}^{n} \binom{n}{r} \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{n-r+\nu} \right.$$

$$\times \left(\frac{1}{2} \overline{\frac{q}{\rho} - 1} \right)^{n-r} \rho^{n-r+\nu/2} \frac{(\sigma-m)!}{(\sigma-m-r+\nu)!}$$

$$\left. \times \left\{ e^{-q(\gamma/u)t/2} \left(\frac{\gamma}{u} t \right)^{\sigma-r+\nu/2-1/2} W_{k+\nu/2,m-\nu/2} \left(\rho \frac{\gamma}{u} t \right) \right\} \right|$$

$$\int_A^\beta \left(\frac{\gamma}{u} \right)^n \left| \phi_n(\eta) \right| d\eta.$$

For $\beta(t, n)$ an alternative expression can be obtained as

$$\beta(t,n) = \left\langle \phi_n(u), e^{-q(\gamma/u)t/2} \left(\frac{\gamma}{u} t \right)^{\sigma-1/2} W_{k,m} \left(\rho \frac{\gamma}{u} t \right) \right\rangle$$

$$= \left\langle (-1)^n \frac{\gamma^{n+3/2}}{\Gamma(\gamma)} \left(\frac{\gamma}{u} \right)^{k-\sigma+1/2} \frac{d^n}{du^n} \left[u^{-2} \left(\frac{\gamma}{u} \right)^{\sigma-k-n-3/2} \phi(u) \right] \right.$$

$$\times e^{-q(\gamma/u)t/2}\left(\frac{\gamma}{u}t\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\Big\rangle$$

$$= (-1)^n\frac{\gamma^{3/2}}{\Gamma(\gamma)}\int_0^\infty\left[u^{-2}\left(\frac{\gamma}{u}\right)^{\sigma-k-n-3/2}\phi(u)\right]\left[\left(\frac{\gamma}{u}\right)^2\frac{d}{d\left(\frac{\gamma}{u}\right)}\right]^n$$

$$\times\left[\left(\frac{\gamma}{u}\right)^{k-\sigma+1/2} e^{-q(\gamma/u)t/2}\left(\frac{\gamma}{u}t\right)^{\sigma-1/2} W_{k,m}\left(\rho\frac{\gamma}{u}t\right)\right] du.$$

First we put $\gamma/u = z \Rightarrow -(\gamma/u^2)\, du = dz$ and again substituting $zt/\gamma = x$, i.e., $z = (\gamma/t)\, x \Rightarrow dz = (\gamma/t)\, dx$,

$$\beta(t,n) = (-1)^n\frac{\gamma}{\Gamma(\gamma)}\int_0^\infty(\gamma x)^{\sigma-k-n-3/2}\left[(\gamma x)^2\frac{d}{d(\gamma x)}\right]^n$$

$$(\gamma x)^k\cdot e^{-q\gamma x/2} W_{k,m}(\rho\gamma x)]\phi\left(\frac{t}{x}\right) dx.$$

From Goldstein [3] we have $(x^2(d/dx))(x^k e^{-x/2}W_{k,m}(x)) = -x^{k+1}\cdot e^{-x/2}$ $W_{k+1,m}(x)$. Making use of this, we get

$$\beta(t,n) = \frac{\gamma}{\Gamma(\gamma)}\int_0^\infty(\gamma x)^{\sigma-3/2} e^{-q\gamma x/2}\, W_{k+n,m}(\rho\gamma x)\phi\left(\frac{t}{x}\right) dx.$$

it is obvious that the support of $\phi(t/x)$ is contained in $t/B \le x \le t/A$; hence,

$$\beta(t,n) = \frac{\gamma}{\Gamma(\gamma)}\int_{t/B}^{t/A}(\gamma x)^{\sigma-3/2} e^{-q\gamma x/2} W_{k+n,m}(\rho\gamma x)\,\phi\left(\frac{t}{x}\right) dx.$$

In proving this it is sufficient to show that as $\gamma\to\infty$, $\zeta(t)\beta(t,n)$ converges to $\zeta(t)\phi(t)$ in $\mathfrak{L}_{a,b}$ for every a and b, where $a < b$. We can prove this by showing that for every choice of the real number h and nonnegative integer r, the function $e^{ht}D_t^r\beta(t,n)$ converges to $e^{ht}D_t^r\phi(t)$ uniformly in $T < t < \infty$:

$$\beta(t,n) = \frac{\gamma}{\Gamma(\gamma)}\int_0^\infty(\gamma x)^{\sigma-3/2} e^{-q\gamma x/2} W_{k,m}(\rho\gamma x)\,\phi\left(\frac{t}{x}\right) dx.$$

Substituting the asymptotic estimate of $W_{k,m}(x)$ for large x (Whittaker and Watson [9]), we find that

$$\begin{aligned}\beta(t,n) &= \frac{\gamma}{\Gamma(\gamma)}\int_0^\infty (\gamma x)^{\gamma-1}e^{-q\gamma x}\,\phi\left(\frac{t}{x}\right)dx\\ &= \frac{\gamma^{\gamma+1}}{\Gamma(\gamma+1)}\int_0^\infty x^{\gamma-1}e^{-q\gamma x}\,\phi\left(\frac{t}{x}\right)dx.\end{aligned}$$

Thus we can write

$$e^{ht}D_t^r[\beta(t,n)-\phi(t)] = e^{ht}\frac{\gamma^{\gamma+1}}{\Gamma(\gamma+1)}\int_0^\infty x^{\gamma}e^{-q\gamma x}\left[x^{-r-1}\phi^{(r)}\left(\frac{t}{x}\right)-\phi^{(r)}(t)\right]dx.$$

It can be estimated by breaking the integral into three integrals from 0 to $1-\delta$, $1-\delta$ to $1+\delta$, and $1+\delta$ to ∞, where $0<\delta<1$. Let I_1, I_2, I_3 be the integrals.

By using the results

$$\begin{aligned}&\frac{\gamma^{\gamma+1}}{\Gamma(\gamma)}\int_{1-\delta}^{1+\delta} x^{\gamma}e^{-q\gamma x}\,dx<1\\ &\frac{\gamma^{\gamma+1}}{\Gamma(\gamma)}\int_0^{1-\delta} x^{\gamma}e^{-q\gamma x}\,dx=O(1)\\ &\frac{\gamma^{\gamma+1}}{\Gamma(\gamma)}\int_{1+\delta}^{\infty} x^{\gamma}e^{-q\gamma x}\,dx=O(1)\end{aligned}$$

as $\gamma\to\infty$ by Zemannian [11] we find that the integrals $|I_1|$, $|I_2|$, and $|I_3|$ are each less than $\varepsilon/3$. This completes the proof.

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